Generic regularity of minimal hypersurfaces in dimension 8

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Joint work with Zhihan Wang

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Outline

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(M^{n+1}, g): $(n+1)$-dimensional closed Riemannian manifold

$\Sigma^n \subset M^{n+1}$: an embedded closed hypersurface

Definition

- Σ is a **minimal hypersurface**, if $\delta\,\mathrm{Area}(\Sigma) = 0 \iff \vec{H} = 0$;
- For a minimal hypersurface Σ, there exists a **Jacobi operator** L_Σ associated to $\delta^2\,\mathrm{Area}(\Sigma)$,
$$L_\Sigma = -\Delta_\Sigma + (|A|^2 + \mathrm{Ric}(\vec{\nu}, \vec{\nu}))\,;$$
- The **Morse index** of a minimal hypersurface Σ is the dimension of the maximal subspace where L_Σ is negatively definite; in particular, Σ is **stable** if $\mathrm{index}(\Sigma) = 0$;
- The minimal hypersurface Σ is **non-degenerate** if 0 is not an eigenvalue of L_Σ.

Note. One can obtain similar definitions for integral currents or integral varifolds.

Preliminaries

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Question: Is there a minimal hypersurface in any given Riemannian manifold (M^{n+1}, g)?

Answer: Yes.

Theorem (E. de Giorgi, Federer-Fleming, J. Simons 60's - 70's)

If $H_n(M^{n+1}) \neq 0$, then there exists a minimal hypersurface Σ^n with

$$\dim(\mathrm{Sing}(\Sigma)) \leq n - 7\,.$$

Note.

- $x \in \mathrm{Reg}(\Sigma)$ if $x \in \overline{\Sigma}$ and there is a neighborhood $x \in U$ such that $\overline{\Sigma} \cap U$ is a smooth hypersurface.
- $\mathrm{Sing}(\Sigma) = \overline{\Sigma} \setminus \mathrm{Reg}(\Sigma)$.
- WLOG, we always assume that $\Sigma = \mathrm{Reg}(\Sigma)$.

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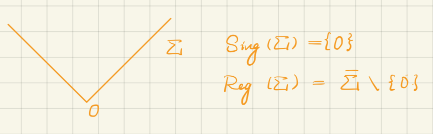

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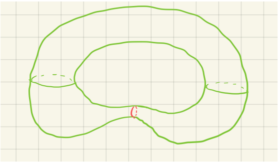
Toy model.



Minimize area among all meridians.

$\xrightarrow{\text{GMT}}$ there exists an area minimizer Σ^n.

$\xrightarrow{\text{Dimension reduction}}$ Σ^n is smooth outside a small set.

Theorem (Almgren-Pitts, Schoen-Simon '81)

For any given closed (M^{n+1}, g), there exists a minimal hypersurface Σ^n with $\dim(\mathrm{Sing}(\Sigma)) \leq n - 7$.

Almgren-Pitts min-max theory

Toy model:
Consider the set $\mathcal{P}$ of all *sweepouts*:

$$\Phi : I \to \mathcal{Z}_n(S^{n+1}, \mathbb{Z}_2) .$$

Define min-max width

$$\mathcal{W} = \inf_{\Phi \in \mathcal{P}} \sup_{t \in I} \mathbf{M}(\Phi(t)) .$$

$\implies \Sigma^n$ realizing the min-max width.

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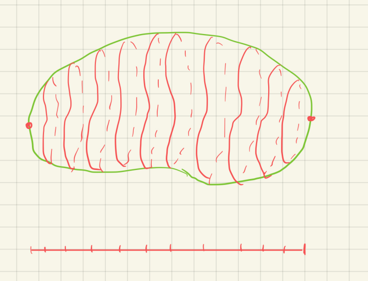

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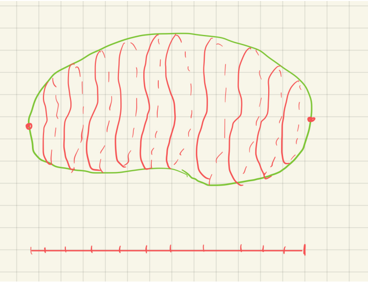

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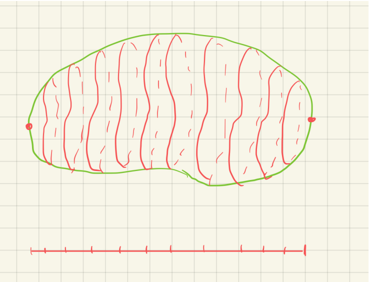

Existence of minimal hypersurfaces

Theorem

For $2 \leq n \leq 6$, there exist ∞ many minimal hypersurfaces.
For $n \geq 7$, this also holds for generic metrics.

Remarks.

1. Marques - Neves '15: $\mathrm{Ric} > 0$;
   Irie-Marques-Neves '17: Generic metrics for $2 \leq n \leq 6$;
   A. Song '18: Any metrics for $2 \leq n \leq 6$;
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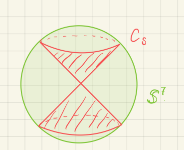
Singularities

Simons Cone



- $C_S \cap S^7 = S^3(\frac{1}{\sqrt{2}}) \times S^3(\frac{1}{\sqrt{2}})$;
- C_S is area-minimizing (Bombieri-de Giorgi-Giusti '69);
- $\dim(\mathrm{Sing}(C_S)) = 0 = 7 - 7$.

Theorem (N. Smale '99)

For any $n \geq 7$, there is (N^{n+1}, g) and a homological area minimizer M^n such that $\mathrm{Sing}(M) = \{2 \text{ points}\}$.

Theorem (L. Simon '21)

In $\mathbb{R}^{n+1+l}(n \geq 7, l \geq 1)$, for any closed subset $K \subset \{0\} \times \mathbb{R}^l$, there exist a metric g C^∞ closed to g_{Eucl} and a strictly stable minimal hypersurface $\Sigma^{n+l} \subset (\mathbb{R}^{n+l}, g)$, such that

$$\mathrm{Sing}\,\Sigma = K. \quad (1)$$

Note. There examples suggest that singularities are inevitable in general.

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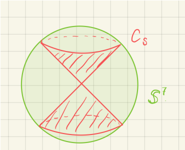

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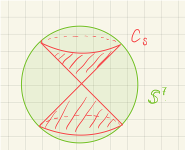

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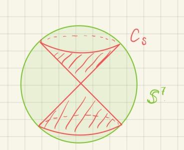

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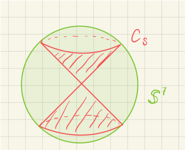

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Generic regularities

Question: Can we perturb away singularities generically? In particular, for any given N^{n+1} with a C^∞-generic metric g, is there a smooth minimal hypersurface?

From now on, we focus on the case $n+1=8$. The higher dimensional cases are widely open.

Theorem (N. Smale '93)

If $H_7(M^8,\mathbb{Z})\neq 0$, then for a generic metric g, there exists a smooth homological area minimizer Σ^7 in (M^8,g).

The proof relies on the analysis of area-minimizing cones C^n with $\mathrm{Sing}(C)=\{0\}$ by Hardt-Simon '85.

Theorem (Hardt-Simon '85)

If E is one of the two components E_+, E_- of $\mathbb{R}^{n+1}\setminus\overline{C}$, then there is an oriented connected real analytic minimzing hypersurface $S\subset E$ such that $\mathrm{dist}(S,0)=1$. Moreover,

1. *$\forall\xi\in E$, the ray $\{\lambda\xi:\lambda>0\}\cap S$ is a transverse point;*
2. *$\mathrm{Sing}(S)=\emptyset$;*
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Question: Can we perturb away singularities generically? In particular, for any given N^{n+1} with a C^∞-generic metric g, is there a smooth minimal hypersurface?
From now on, we focus on the case $n+1=8$. The higher dimensional cases are widely open.

Theorem (N. Smale '93)

If $H_7(M^8,\mathbb{Z})\neq 0$, then for a generic metric g, there exists a smooth homological area minimizer Σ^7 in (M^8,g).

The proof relies on the analysis of area-minimizing cones C^n with $\mathrm{Sing}(C)=\{0\}$ by Hardt-Simon '85.

Theorem (Hardt-Simon '85)

If E is one of the two components E_+, E_- of $\mathbb{R}^{n+1}\setminus\overline{C}$, then there is an oriented connected real analytic minimzing hypersurface $S\subset E$ such that $\mathrm{dist}(S,0)=1$. Moreover,

1. *$\forall\xi\in E$, the ray $\{\lambda\xi:\lambda>0\}\cap S$ is a transverse point;*
2. *$\mathrm{Sing}(S)=\emptyset$;*
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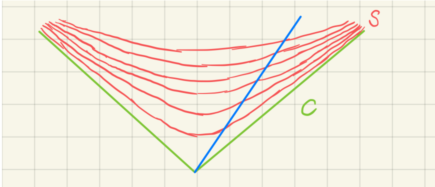

Figure: Hardt-Simon "foliation"

"Proof" of N. Smale '93:

1. Start with (M^8, g_0) and an area-minimizer Σ^7.
2. For any $p \in \mathrm{Sing}(\Sigma)$, using Hardt-Simon, we can concatenate $\Sigma^7 \setminus B_{r_p}(p)$ and $\lambda S \cap B_{r_p}(p)$ and obtain $\tilde{\Sigma}$ smooth.
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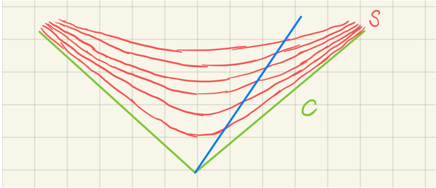

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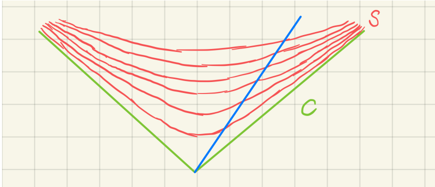

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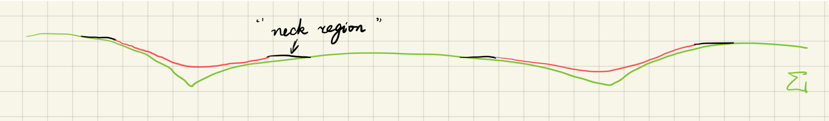

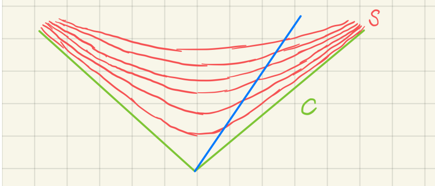

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For any metric, there exists a minimal hypersurface Σ *such that*

$$\mathrm{index}(\Sigma) + \mathcal{H}^0(\mathrm{Sing}_{nm}(\Sigma)) \leq 1\,.$$

In particular, if $\mathrm{Ric} > 0$*, then for a generic metric,* Σ *is smooth.*

Theorem (L.-Wang '20)

For a generic metric g, there is a smooth minimal hypersurface $\Sigma^7 \subset (M^8, g)$.

Main difficulty: Hardt-Simon only works for area-minimizing cones, not for stable but not area-minimizing cones. Thus, we need a global perturbation instead of a local one.

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Outline

Proposition (Wang '20)

Given (M^8, g), $\Sigma^7 \subset M^8$ minimal and nondegenerate, a generic function $f \in C^\infty(M)$ and $c_j \to 0$, if in each $(M^8, (1 + c_j f)g)$, there exists a minimal hypersurface Σ_j satisfying that

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Then there exist $p \in \mathrm{Sing}(\Sigma)$ and $r_p > 0$, such that for infinitely many j,

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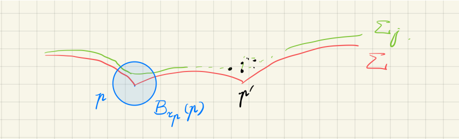

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1. By Schoen-Simon '81, near $\mathrm{Reg}(\Sigma)$, Σ_j is smooth;
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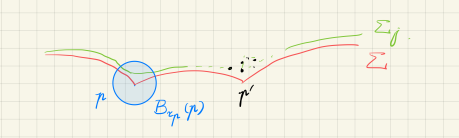

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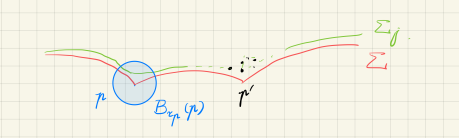

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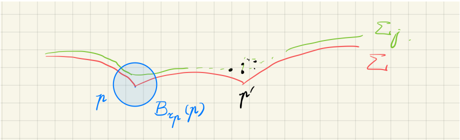

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Uniqueness of min-max width realization

Recall

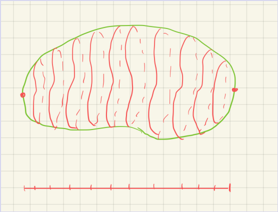
Consider the set $\mathcal{P}$ of all *sweepouts*:

$$\Phi : I \to \mathcal{Z}_n(S^{n+1}, \mathbb{Z}_2)\,.$$

Define min-max width

$$\mathcal{W} = \inf_{\Phi \in \mathcal{P}} \sup_{t \in I} \mathbf{M}(\Phi(t))\,.$$

$\Longrightarrow$ Σ^n realizing the min-max width.



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Unique realization holds under some technical assumptions.

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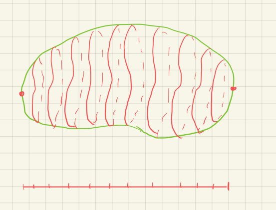

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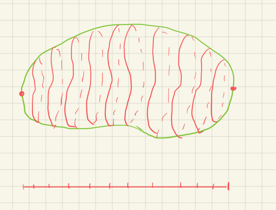

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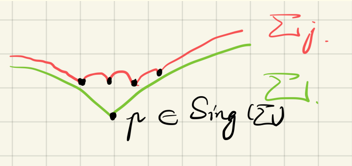

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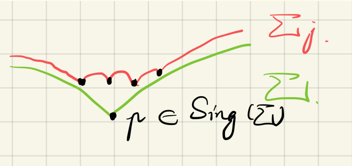

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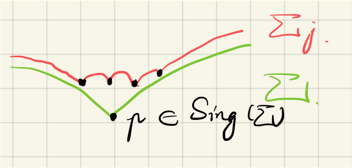

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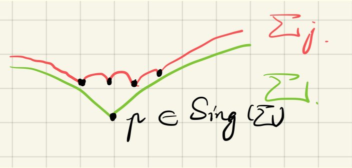

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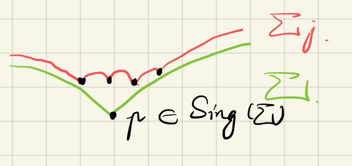

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Outline

Further discussion

Open Questions:

1. Can the perturbation work for multi-parameter min-max minimal hypersurfaces in dimension 8?
2. How about higher dimensions?

For the first question, it seems possible, if we can generalize a structure theorem by Brian White to singular minimal hypersurfaces.

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A recent work by Nick Edelen has made an important step towards it in some perspective, but more work needs to be done.

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In a closed (N^8, g), the number of diffeomorphism classes of minimal hypersurfaces of bounded area and Morse index is finite.

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1. Can the perturbation work for multi-parameter min-max minimal hypersurfaces in dimension 8?
2. How about higher dimensions?

For the first question, it seems possible, if we can generalize a structure theorem by Brian White to singular minimal hypersurfaces.

Theorem (B. White '91)

For generic metrics, smooth minimal hypersurfaces are non-degenerate.

A recent work by Nick Edelen has made an important step towards it in some perspective, but more work needs to be done.

Theorem (N. Edelen '21)

In a closed (N^8, g), the number of diffeomorphism classes of minimal hypersurfaces of bounded area and Morse index is finite.

The second one seems beyond our reach so far, since not so much is known about singular sets except isolated singular points.

Thank you