INTRODUCTION TO ALMGREN-PITTS MIN-MAX THEORY (PRELIMINARY VERSION)

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ABSTRACT. These notes[1] are for a 5-day mini-course that I gave at Tsinghua University starting from July 1, 2019. In the course, we shall present the idea on how to search for minimal hypersurfaces in a closed manifold using Almgren-Pitts min-max theory.

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[1] Errors, typos or inaccuracies might occur in the notes. I will appreciate it if someone spotting them could let me know at yl15@math.princeton.edu.

1. Introduction

In Riemannian geometry, the existence and regularity of minimal hypersurfaces is one of the central problems. In 1982, motivated by the existence results in $(n+1)$-dimensional closed manifolds by G.D. Birkhoff ([Bir17], $n=1$), J. Pitts ([Pit81], $2 \leq n \leq 5$) and R. Schoen and L. Simon ([SS81], $n \geq 6$), S.-T. Yau proposed the conjecture of existence of infinitely many minimal surfaces in 3-dimensional Riemannian manifolds.

Conjecture 1 (Yau's conjecture, [Yau82])**.** *Any closed three-dimensional manifold must contain an infinite number of immersed minimal surfaces.*

In [IMN18], K. Irie, F.C. Marques and A. Neves, using the Weyl law [LMN18] for volume spectra by Y. Liokumovich and the last two named authors, proved a stronger version of Yau's conjecture in the generic case.

Theorem 1.1 (Density of minimal hypersurfaces in the generic case, [IMN18])**.** *Let M^{n+1} be a closed manifold of dimension $n+1$, with $3 \leq n+1 \leq 7$. Then for a C^∞-generic Riemannian metric g on M, the union of all closed, smooth, embedded minimal hypersurfaces is dense.*

Later, in [MNS17], F.C. Marques, A. Neves and A. Song gave a quantitative description of the density, i.e., the equidistribution of a sequence of minimal hypersurfaces under the same condition.

The Yau's conjecture for $2 \leq n \leq 6$ for general C^∞ metrics was finally resolved by A. Song [Son18] using the methods developed by F.C. Marques and A. Neves in [MN17].

Recently, X. Zhou [Zho19] confirmed Marques-Neves multiplicity one conjecture for bumpy metrics, which combined with work of Marques-Neves [MN18] on the Morse index leads to:

Theorem 1.2 (Theorem 8.4, [MN18])**.** *Let g be a C^∞-generic (bumpy) metric on a closed manifold M^{n+1}, $3 \leq (n+1) \leq 7$. For each $k \in \mathbb{N}$, there exists a smooth, closed, embedded, multiplicity one, two-sided, minimal hypersurface Σ_k such that*

$$\omega_k(M,g) = \mathrm{area}_g(\Sigma_k) \quad \mathrm{index}(\Sigma_k) = k \tag{1.1}$$

and

$$\lim_{k \to \infty} \frac{\mathrm{area}_g(\Sigma_k)}{k^{\frac{1}{n+1}}} = a(n)\mathrm{vol}(M,g)^{\frac{n}{n+1}} \tag{1.2}$$

where $a(n) > 0$ is a dimensional constant in Weyl law.

Note that most of the results above were obtained in the Almgren-Pitts min-max setting (Zhou's result on the multiplicity one conjecture was based on a new regularization of the area functional in Cacciopoli min-max setting developed by him and J. Zhu [ZZ18]). In the Allen-Cahn min-max setting, P. Gaspar and M.A.M. Guaraco [GG18] and O. Chodosh and C. Mantoulidis [CM18]($n=2$) gave similar results. In particular, O. Chodosh and C. Mantoulidis proved the multiplicity one conjecture in 3-manifolds before Zhou's result.

More recently, the author adapted Irie-Marques-Neves' approach to prove Yau's conjecture in higher dimensions ($n \geq 7$) for generic metrics.

Theorem 1.3 (Theorem 1.3, [Li19])**.** *Given a closed manifold $M^{n+1}(n \geq 7)$, there exists a (Baire sense) generic subset of C^∞ metrics such that M endowed with any* *one of those metrics contains infinitely many singular minimal hypersurfaces with optimal regularity.*

Remark 1. *In fact, for $(n+1) \leq 8$, the adapted method in the proof implies that for generic metrics, the p-widths minimal hypersurfaces are dense, which supports the equidistribution of p-widths minimal hypersurfaces conjectured by F.C. Marques and A. Neves.*

In a working project, the author is also able to prove the Morse index upper bound for minimal hypersurface from Almgren-Pitts theory without dimensional restriction, which implies the following theorem:

Theorem 1.4. *Suppose that (M^{n+1}, g) is a closed Riemannian manifold ($n+1 \geq 3$). Then for any $p \in \mathbb{N}^+$, there exists a stationary, integral varifold V with $\mathrm{spt}(V) = \Sigma$ such that*

- $\|V\|(M) = \omega_p(M, g)$
- $\mathrm{Ind}(\Sigma) \leq p$
- $\mathcal{H}^s(\mathrm{sing}(\Sigma)) = 0, \forall s > n-7$. *In particular, when $n \leq 6$, Σ is smooth.*

Remark 2. *In a Riemann surface, the index upper bound is still true but the support of the varifold might only be a geodesic network rather than a closed geodesic.*

Outline of the Notes. In Section 2, we shall recall some definitions and notations from geometric measure theory, and introduce the notion 'Almost Minimizing' which helps to establish the regularity result in Almgren-Pitts theory.

In Section 3, we shall see an surprising and useful result on the topology of the space of modulo 2 flat chains proved by F.J. Almgren in his PhD thesis. The nontrivial topology of this space enable us to apply Morse theory in order to search critical points, i.e., minimal hypersurfaces.

In Section 4, the existence result of a non-empty almost minimizing varifold in arbitrary closed manifolds will be proved following J. Pitts' idea. The setup of his min-max theory and his novel combinatorial arguments with some sort of simplification will be presented in this section.

In Section 5, instead of following Pitts' original idea involving Schoen-Simon-Yau curvature estimate, we will outline the general result by R. Schoen and L. Simon so as to prove the regularity of almost minimizing varifolds in arbitrary dimesional closed manifolds. As a consequence of Section 4 and Section 5, we obtain one minimal hypersurface in a closed manifold, possibly with a small singularity set in higher dimesions.

In Section 6, we will outline the proof for Yau's conjecture in generic metrics (in arbitrary dimensions).

2. Preliminaries

2.1. Definitions and Notations.

In this part we recall some definitions from geometric measure theory. Readers who are not familiar with these are recommended to refer to the standard lecture notes of L. Simon [Sim84] and the classic book of H. Federer [Fed96].

Let (M^{n+1}, g) be a closed Riemannian manifold isometrically embedded in $\mathbb{R}^L$.

In general, the Almgren-Pitts min-max theory will work simultaneously on both the space of currents and the space of varifolds. The basic notations for these spaces are:

- $\mathbf{I}_k(M)$: the space of k-dimensional integral currents in $\mathbb{R}^L$ supported in M
- $\mathcal{Z}_k(M)$: the space of $T \in \mathbf{I}_k(M)$ with $\partial T = 0$
- $\mathbf{I}_k(M;\mathbb{Z}_2)$: the space of k-dimensional mod 2 flat chains in $\mathbb{R}^L$ supported in M (See 4.4.26 [Fed96])
- $\mathcal{Z}_k(M;\mathbb{Z}_2)$: the space of $T \in \mathbf{I}_k(M;\mathbb{Z}_2)$ with $\partial T = 0$
- $\mathcal{Z}_k(M,K;\mathbb{Z}_2)$; the space of $T \in \mathbf{I}_k(M;\mathbb{Z}_2)$ with $\operatorname{spt}(\partial T) \subset K$
- $\mathcal{V}_k(M)$: the closure, in the weak topology, of the space of k-dimensional rectifiable varifolds in $\mathbb{R}^L$ supported in M
- $\mathcal{IV}_k(M)$: the space of k-dimemsional integral rectifiable varifolds

For $T \in \mathbf{I}_k(M)$ or $\mathbf{I}_k(M;\mathbb{Z}_2)$, the associated integral varifold and Radon measure in M are denoted by $|T|$ and $\|T\|$ respectively, and for $V \in \mathcal{V}_k(M)$, the associated Radon measure is denoted by $\|V\|$.

Now, let's consider some metrics related to the topology of the spaces above.

On $\mathcal{V}_k(M)$, we can define the **F metric** as in [Pit81],

$$\begin{aligned} &\mathbf{F} : \mathcal{V}_k(M) \times \mathcal{V}_k(M) \to \overline{\mathbb{R}_+} \\ &\mathbf{F}(V,W) = \sup\{V(f) - W(f) : f \in C_c(\mathrm{G_k}(M)), \\ &\qquad\qquad \text{where } |f| \leq 1, \operatorname{Lip}(f) \leq 1\} \end{aligned} \tag{2.1}$$

Note that for any $c > 0$, the $\mathbf{F}$-metric topology and the weak topology coincide on $\mathcal{V}_k(M) \cap \{V : \|V\|(M) \leq c\}$. The restriction of F to a borel set $B \subset M$ will be defined as

$$\mathbf{F}_B(V,W) := \mathbf{F}_B(V \llcorner \mathrm{G_k}(B), W \llcorner \mathrm{G_k}(B)) \tag{2.2}$$

On $\mathbf{I}_k(M)$, we can define the **mass norm M**, the **flat metric** $\mathcal{F}_M$ and the **F metric** as follows.

- For any $T \in \mathbf{I}_k(M)$,

$$\mathbf{M}(T) = \sup_{\|\omega\| \leq 1, \omega \in \mathcal{D}^n(M)} T(\omega) \tag{2.3}$$

- We define $\mathcal{F}_M$ as

$$\begin{aligned} &\mathcal{F}_M : \mathbf{I}_k(M) \times \mathbf{I}_k(M) \to \overline{\mathbb{R}^+} \\ &\mathcal{F}_M(T_1,T_2) = \inf\{\mathbf{M}(R) + \mathbf{M}(S) : T_1 - T_2 = \partial R + S, \\ &\qquad\qquad \text{where } R \in \mathbf{I}_{k+1}(M), S \in \mathbf{I}_k(M)\} \end{aligned} \tag{2.4}$$

- The $\mathbf{F}$ metric is defined as

$$\begin{aligned} &\mathbf{F} : \mathbf{I}_k(M) \times \mathbf{I}_k(M) \to \overline{\mathbb{R}_+} \\ &\mathbf{F}(T,S) = \mathbf{F}(|T|,|S|) + \mathcal{F}_M(T,S) \end{aligned} \tag{2.5}$$

We assume that $\mathbf{I}_k(M)$ and $\mathcal{Z}_k(M)$ are endowed with the flat metric, while $\mathbf{I}_k(M;\nu)$ and $\mathcal{Z}_k(M;\nu)$ denote the same sets endowed with the metric ν ($\mathbf{M}$ norm or $\mathbf{F}$ metric).

Note that all the metrics could be defined on $\mathbf{I}_k(M;\mathbb{Z}_2)$ and we also have similar notations $\mathbf{I}_k(M;\nu;\mathbb{Z}_2)$ and $\mathcal{Z}_k(M;\nu;\mathbb{Z}_2)$.

At a first glance, the limiting current of integral rectifiable currents and the limiting varifold of their associated integral varifolds might be related. However, they could be rather different.

Example 2.1. *Let $S^n_r \subset \mathbb{R}^{n+1}$ be a n-dimensional sphere with radius r centered at 0 and $[[S^n_r]]$ be the associated integral current whose orientation is given by the outward normal. Consider a sequence of integral currents*

$$T_i = [[S^n_{1+1/i}]] - [[S^n_1]] \tag{2.6}$$

Obviously, in the flat topology, $T_i \to 0$ while in the weak topology, $|T_i| \to 2|[[S^n_1]]|$.

Fortunately, we have the following fact.

Exercise 1 (2.18(f) [Pit81])**.** *Let $T, T_1, T_2, \cdots \in \mathbf{I}_k(M)$, $\lim_i T_i = T$ in the flat topology, and $\lim_i |T_i| = V \in \mathcal{V}_k(M)$ in the weak topology. Then $\|T\| \le \|V\|$ and the following three conditions are equivalent:*

1. $\|T\| = \|V\|$
2. $|T| = V$
3. $\|V\|(M) = \mathbf{M}(T)$

Note that in the general case, we could not concluded that $|T| \le V$ as one would expect due to the following counter example.

Example 2.2. *Let T_i be the integral currents associated to the curves with endpoints $(0,0)$ and $(1,0)$ in **Figure 1**. T_i converges to the current associated to the interval $[0,1]$, but $|T_i|$ would converge to a varifold which is not even rectifiable. More precisely, the limit of $|T_i|$ would be the sum of two product Radon measures, i.e., the products of Lebesgue measure on $[0,1]$ and δ measures on $G(2,1)$ corresponding to the lines making 45° with the $[0,1]$ interval.*

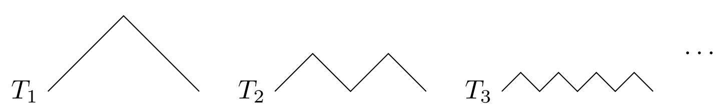

FIGURE 1. Example 2.2

Exercise 2. *Use **Exercise 1** to prove the following fact: If $S, T_1, T_2, \cdots \in \mathcal{Z}_k(M)$, then*

$$\lim_{i\to\infty} \mathbf{F}(T_i, S) = 0 \tag{2.7}$$

if and only if

$$\lim_{i\to\infty} \mathcal{F}(T_i, S) = 0 \text{ and } \lim_{i\to\infty} \mathbf{M}(T_i) = \mathbf{M}(S) \tag{2.8}$$

2.2. Almost Minimizing Varifolds.

Definition 2.1 (Definitions 3.1 [Pit81])**.** *Assume that ν is one of the three metrics $\mathcal{F}_M$, $\mathbf{F}$ and $\mathbf{M}$, and U is an open subset of M.*

(1) *For each pair of numbers $\varepsilon, \delta > 0$, we define*

$$\mathfrak{a}_k(U; \varepsilon, \delta; \nu)$$

to be the set of all currents $T \in \mathcal{Z}_k(M, M\backslash U; \mathbb{Z}_2)$ with the following property:

If a finite sequence $\{T_i\}_{i=0}^q$ in $\mathcal{Z}_k(M, M\backslash U; \mathbb{Z}_2)$ with

- *$T_0 = T$ and $\operatorname{spt}(T - T_i) \subset U, \forall i$*

- $\nu(T_i - T_{i-1}) \le \delta$ *for all* $i \ge 1$
- $\mathbf{M}(T_i) \le \mathbf{M}(T) + \delta$

then $\mathbf{M}(T) - \mathbf{M}(T_q) \le \varepsilon$.

(2) *We say that* $V \in \mathcal{V}_k(M)$ *is* ν ***almost minimizing*** *in* U *if and only if for each positive* ε *there exist a* $\delta > 0$ *and* $T \in \mathfrak{a}(U; \varepsilon, \delta; \nu)$ *such that* $\mathbf{F}_U(V, |T|) < \varepsilon$. *We say that* V *is* ν ***almost minimizing at*** p *if there exists a neighborhood* U *of* p *such that* V *is* ν *almost minimizing inside* U. *We would omit* ν *if* $\nu = \mathcal{F}_M$.

Theorem 2.1 (Theorem 3.3 [Pit81])**.** *If* $V \in \mathcal{V}_k(M)$ *is almost minizing in* U*, then* V *is stable in* U.

Exercise 3. *Prove the theorem by contradiction.*

Theorem 2.2 (Theorem 3.9 [Pit81])**.** *Let* $V \in \mathcal{V}_k(M)$.

(1) *Each of the these statements implies the one following it.*
  (a) V *is almost minimizing in* U.
  (b) V *is* $\mathbf{F}$ *almost minimizing in* U.
  (c) V *is* $\mathbf{M}$ *almost minimizing in* U.
  (d) V *is almost minimizing in any relatively open subset* W *of* M *with* $W \subset\subset U$.

(2) *The following statements are equivalent:*
  (a) V *is almost minimizing at* p.
  (b) V *is* $\mathbf{F}$ *almost minimizing at* p.
  (c) V *is* $\mathbf{M}$ *almost minimizing at* p.

The proof of the theorem is not difficult but rather lengthy, so we will not present it here. The following simple exercise could be viewed as a hint for the proof.

Exercise 4. *On* $\mathbb{R}^2$*, construct a* $\mathbf{M}$*-continuous path between two currents* $[0,1] \times 0 \cup 0 \times [0,1]$ *and* $[0,1] \times 1 \cup 1 \times [0,1]$ *fixing the boundary.*

For a local almost minimizing codimension 1 varifold V, we could locally construct a class of comparison surfaces to V in $\mathcal{V}_k(M)$, which would be helpful in proving the rectifiability and even the smoothness of V (Consruction 3.10 [Pit81]).

Let K be a compact subset of U and $V \in \mathcal{V}_k(M)$ be almost minimizing in U. We construct

$$\mathfrak{b}(V; U, K) \tag{2.9}$$

a class of comparison surfaces to V in $\mathcal{V}_k(M)$ from the definition of the almost minimizingness.

Note that there exist sequences $\{\delta_i\}, \{\varepsilon_i\}$ of positive real numbers with $\delta_i \searrow 0, \varepsilon_i \searrow 0$ and a sequence $T_i \in \mathfrak{a}_k(U; \varepsilon_i, \delta_i)$ with $\mathbf{F}_U(V, |T_i|) < \varepsilon_i$. Now we can fix the integer i and define μ_i to be the infimum of all numbers $\mathbf{M}(S)$ corresponding to all S for which there exists a sequence $T_i = T_i^1, T_i^2, \cdots T_i^q = S$ in $\mathcal{Z}_k(M, M \backslash U; \mathbb{Z}_2)$ with

$$\begin{aligned} &\operatorname{spt}(T_i^j - T_i) \subset K \\ &\sup_j \mathbf{M}(T_i^j) \le \mathbf{M}(T_i) + \delta_i \\ &\sup_j \mathcal{F}_M(T_i^j - T_i^{j-1}) \le \delta_i \end{aligned} \tag{2.10}$$

Then we can choose a finite sequence $T_i = T_i^1, T_i^2, \cdots, T_i^q = T_i^*$ in $\mathcal{Z}_k(M, M\backslash U; \mathbb{Z}_2)$ with $\mathbf{M}(T_i^*) = \mu_i$.

Define

$$V_i^* = V \llcorner G_k(M\backslash U) + |T_i^*| \llcorner G_k(U) \tag{2.11}$$

and

$$V^* = \lim_i V_i^* \tag{2.12}$$

The set $\mathfrak{b}(V; U, K)$ consists of all such V^*, which is compact and nonempty.

Theorem 2.3 (Theorem 3.11 [Pit81])**.** *Suppose $V \in \mathcal{V}_k(M)$, V is almost minimizing in U, K is a compact subset of U, and $V^* \in \mathfrak{b}(V; U, K)$. Then*

1. *$V \llcorner G_k(M\backslash K) = V^* \llcorner G_k(M\backslash K)$.*
2. *V^* is almost minimizing in U.*
3. *$\|V\|(M) = \|V^*\|(M)$.*
4. *For each $\varepsilon > 0$, there exists $T \in \mathcal{Z}_k(M, M\backslash U; \mathbb{Z}_2)$ such that $\mathbf{F}_U(V^*, T) < \varepsilon$ and $T \llcorner Z$ is locally area minimizing with respect to $(Z, \emptyset)$ for all compact Lipschitz neighborhood retracts $Z \subset \text{Int}(K)$. In addition, $|T|$ can be chosen to be stable.*
5. *$V^* \in \mathcal{IV}_k(\text{Int}(K))$.*

Proof. The first four arguments follow from the construction. To prove (5), it suffices to show that $V^* \llcorner \mathrm{G_k}(Z) \in \mathcal{IV}_k(Z)$ whenver Z is relatively open subset of M with $\overline{Z} \subset \text{Int}(K)$ and $\|V^*\|(\partial Z) = 0$.

Letting $V_1^*, V_2^*, \cdots$ in the definition of V^*, then we have

$$\lim_i V_i^* \llcorner \mathrm{G_k}(Z) = V^* \llcorner \mathrm{G_k}(Z) \tag{2.13}$$

$$V_i^* \llcorner \mathrm{G_k}(Z) \in \mathcal{IV}_k(Z) \tag{2.14}$$

$$V_i^* \llcorner \mathrm{G_k}(Z) \text{ is stationary in } Z \tag{2.15}$$

$$V^* \llcorner \mathrm{G_k}(Z) \in \mathcal{IV}_k(Z) \tag{2.16}$$

□

Theorem 2.4 (Theorem 3.13 [Pit81])**.** *Let $V \in \mathcal{V}_k(M)$. If for each $p \in M$, there exists a fininte positive number r with the property that V is almost minimizing in $A(p, s, r)$ (an open annulus centered at p with inner and outer radii s and r, respectively) for all $0 < s < r$, and if V is stationary, then $V \in \mathcal{IV}_k(M)$.*

Outline of Proof. Observe that V is almost minimizing almost everywhere.

The first step is to show the rectifiability of V. In fact, we know that at each point p where V is almost minimizing, the replacement $V^* \in \mathcal{IV}_k(M)$ which would help to induce the positive lower bound of the density of V, $\Theta^k(\|V\|, p)$. Allard's regularity theorem would then imply the rectifiability of V.

To show that $\Theta(V, p)$ is an integer, we could restrict to the point p where V has a unique approximate tangent space T_pV. At those points, we could use a sequence of scaled V^* to approximate T_pV. The Federer-Fleming compactness for stationary varifolds would give the integrability of T_pV as well. □

3. Almgren's Isomorphism

In 1962, in his PhD thesis, F.J. Almgren showed the following isomorphism.

Theorem 3.1 (Theorem (7.5) [Alm62])**.** *For any smooth closed Riemannian manifold M and any nonnegative integers k and m, we have*

$$\pi_k(\mathcal{Z}_m(M);0) \cong H_{k+m}(M) \tag{3.1}$$

The idea of the proof is to use the isoperimetric choice from isoperimetric theorem to construct a canonical map $F_M : \pi_k(\mathcal{Z}_m(M);0) \to H_{k+m}(M)$ and then we need to show that it is both surjective and injective.

Let's first start with the isoperimetric theorem.

Theorem 3.2 ([Sim84])**.** *Suppose $T \in \mathcal{Z}_k(\mathbb{R}^L)(k \geq 1, L > k)$, $\operatorname{spt}T$ is compact. Then there exists $R \in \mathbf{I}_{k+1}(\mathbb{R}^L)$ with $\operatorname{spt}R$ compact, $\partial R = T$ and*

$$\mathbf{M}(R)^{\frac{k}{k+1}} \leq c\mathbf{M}(T) \tag{3.2}$$

where $c = c(k, L)$.

In general, the theorem is not true if restricted to a closed manifold whose homology is nontrivial. However, for a fixed manifold M, if $\mathbf{M}(T)$ is small enough, it is still true as follows.

Proposition 3.1 (Proposition (1.1) [Alm62])**.** *For each smooth closed manifold M isometrically embedded in $\mathbb{R}^L$, there are numbers $\varepsilon > 0$ and $c < \infty$ such that, if $T \in \mathcal{Z}_k(M), k > 0$ and $\mathbf{M}(T) < \varepsilon$, then there exists $R \in \mathbf{I}_{k+1}(M)$ with $\partial R = T$, and $\mathbf{M}(R)^{\frac{k}{k+1}} \leq c\mathbf{M}(T)$*

$$\mathbf{M}(R) \leq \inf\{\mathbf{M}(Q) : Q \in \mathbf{I}_{k+1}(M) \text{ and } \partial Q = T\} \tag{3.3}$$

Moreover, if the smallness condition is replaced by $\mathcal{F}_M(T, 0) < \varepsilon$ for some other small $\varepsilon > 0$, then such an R still exists and $\mathbf{M}(R) = \mathcal{F}_M(T, 0)$.

Proof. In the case where $\mathbf{M}(T) < \varepsilon$, it suffices to find one Q satisfing the other estimates, and then the Federer-Fleming compactness will imply the existence of R.

Since M is a smooth closed submanifold, there exists $\eta > 0$ such that we can construct a Lipschitz retraction r from η-neighborhood $B_\eta(M)$ to M itself.

By the scaled version of Deformation theorem ([Sim84]), we can find an integer linear combination P of disjoint k-dimensional cubes with side length $= \eta/(10\sqrt{L})$ contained in $B_\eta(M)$ so that

$$T = [[P]] + \partial S \tag{3.4}$$

where S is also an integral current supported in $B_\varepsilon(M)$. Moreover, we also have the estimates

$$\begin{aligned} \mathbf{M}(S)^{\frac{k}{k+1}} &\leq c\mathbf{M}(T) \\ \mathbf{M}(P) &\leq c\mathbf{M}(T) \end{aligned} \tag{3.5}$$

Note that if P is nonempty, then there is at least one cube in P so $\mathbf{M}([[P]]) \geq \Big(\eta/(10\sqrt{L})\Big)^k$. Therefore, if $\varepsilon > 0$ is chosen small enough, i.e., $c\varepsilon < \Big(\eta/(10\sqrt{L})\Big)^k$, then $P = \emptyset$. In this case, using the retraction, we obtain that

$$T = r_\# T = \partial\left(r_\# S\right) =: \partial Q \tag{3.6}$$

and the estimate

$$\mathbf{M}(Q) \le \mathrm{Lip}(r)^{k+1}\mathbf{M}(S) \tag{3.7}$$

In sum, we can conclude the existence of one satisfactory Q.

In the case where $\mathcal{F}_M(T,0) < \varepsilon$, by definition, we know that there exists $R \in \mathbf{I}_{k+1}(M)$ and $S \in \mathbf{I}_k(M)$ such that $T = \partial R + S$ and $\mathcal{F}_M(T,0) = \mathbf{M}(R) + \mathbf{M}(S)$. We only need to show that $S = 0$.

Suppose not, since $\mathbf{M}(S) \le \varepsilon$, by the first argument, there exists $R' \in \mathbf{I}_{k+1}(M)$ with $\mathbf{M}(R') = c^{\frac{k+1}{k}}\mathbf{M}(S)^{\frac{k+1}{k}} < \mathbf{M}(S)$ as long as ε is small enough. Now we can instead write $T = \partial(R + R')$ with $\mathbf{M}(R + R') < \mathbf{M}(R) + \mathbf{M}(S)$ which gives a contradiction. □

Definition 3.1. *If T and R are as in the proposition above, then R is called an **isoperimetric choice** for T.*

3.1. **Special Case: Isomorphism for Mod 2 Hypercycles.**

For mod 2 hypercycles $\mathcal{Z}_n(M^{n+1};\mathbb{Z}_2)$, F.C. Marques and A. Neves gave a simpler proof [MN18]. Let's first recall the constancy theorem.

Theorem 3.3 (Constancy Theorem)**.** *If $R \in \mathcal{Z}_{n+1}(M;\mathbb{Z}_2)$ then $R = 0$ or M.*

Corollary 3.1. *The boundary map $\partial : \mathbf{I}_{n+1}(M^{n+1};\mathbb{Z}_2) \to \mathcal{Z}_n^0(M^{n+1};\mathbb{Z}_2)$ is a 2-cover.*

Using Constancy Theorem and Isoperimetric Theorem, we can prove the following lifting property.

Exercise 5 (Lifting Property)**.** *For every continous map $\Psi : I^p \to \mathcal{Z}_n^0(M;\mathbb{Z}_2)$ and $U_0 \in \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ with $\partial U_0 = \Psi(0)$, there exists a unique continuous map $U : I^p \to \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ such that $\partial U(x) = \Psi(x)$ and $U(0) = U_0$.*

On M^{n+1}, we can define a Morse function $f : M \to [0,1]$, and it is easy to see that for any $R \in \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$,

$$t \in [0,1] \to R \cap \{f \le t\} \tag{3.8}$$

is continuous in the flat norm. With f, we can show:

Lemma 3.1. *$\mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ is contractible.*

Now, we can get a short proof of Almgren's Isomorphism in this case.

Proposition 3.2. *$\pi_k(\mathcal{Z}_n(M^{n+1};\mathbb{Z}_2),0) = 0, \forall k \ge 2$.*

Proof. Let $\Psi : I^k \to \mathcal{Z}_n$ with $\Psi(\partial I^k) = 0$. Using the lifting property, we can obtain the lifting map $U : I^k \to \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ with $U(0) = 0$.

Note that ∂I is connected, the constancy theorem gives that $U(\partial I) = 0$ so we can define a homotopy map

$$\begin{aligned} h : I^k \times [0,1] &\to \mathcal{Z}_n^0 \\ h(x,t) &= \partial(U(x) \cap \{f \le t\}) \end{aligned} \tag{3.9}$$

which implies that Ψ is homotopic to 0. □

Proposition 3.3. *$\pi_1(\mathcal{Z}_n(M^{n+1};\mathbb{Z}_2),0) = \mathbb{Z}_2$.*

Proof. Similar to the proof above, let $\Psi : I \to \mathcal{Z}_n$ with $\Psi(\partial I) = 0$ and its lifting $U : I \to \mathbf{I}_{n+1}$ with $U(0) = 0$.

There are two cases.

If $U(1) = 0$, using the same argument above, we can show that Ψ is weakly homotopic to 0.

If $U(1) = M$, say, $\Psi(t) = \partial\{f \le t\}$, it suffices to show that $\Psi \not\sim 0$.

Suppose not, we will have a relative homotopy map $h : I \times I \to \mathcal{Z}_n^0$ with $h(\cdot, 0) = 0$ and $h(\cdot, 1) = \Psi$. Apply the lifting property to h, the constancy theorem and the fact that $U(0) \neq U(1)$ for Ψ would give a contradiction. □

3.2. **Sequences of Chain Maps.**

Definition 3.2 (Definitions (2.1) [Alm62])**.**

(1) *For each* $n = 0, 1, 2, \cdots$, *let* $I(1, n)$ *be the cell complex of the unit interval* $I = [0, 1]$ *whose* 1*-cells are the subintervals,*

$$[0, 1 \cdot 3^{-n}], [1 \cdot 3^{-n}, 2 \cdot 3^{-n}], \cdots, [(3^n - 1) \cdot 3^{-n}, 1]$$

and whose 0*-cells are the endpoints,*

$$[0], [1 \cdot 3^{-n}], \cdots, [1]$$

(2) *For each* $m = 1, 2, 3, \cdots$ *and* $n = 0, 1, 2, \cdots$,

$$I(m, n) = I(1, n)^{m\otimes} = I(1, n) \otimes \cdots \otimes I(1, n)$$

and $I(m, n)_p$ *is the set of* p*-cells in* $I(m, n)$*.*

Definition 3.3 (Definitions (2.3) [Alm62])**.** *A* ***chain map*** $I(m, n) \to \mathbf{I}_*(M)$ ***of degree*** k *is a graded homomorphism*

$$\phi_M : I(m, n) \to \mathbf{I}_*(M) \quad \text{of degree } k \tag{3.10}$$

such that

$$\partial \circ \phi_M(\alpha^p) = \phi_M(\partial \alpha^p) \quad \forall \alpha^p \in I(m, n)_p \tag{3.11}$$

If ϕ_M^0 *and* ϕ_M^1 *are two chain maps of degree* k*, a* ***chain homotopy*** *between* ϕ_M^0 *and* ϕ_M^1 *is a graded homomorphism*

$$\psi_M : I(m, n) \to \mathbf{I}_*(M) \quad \text{of degree } k + 1 \tag{3.12}$$

such that

$$\psi_M(\partial \alpha^p) + \partial \circ \psi_M(\alpha^p) = \phi_M^1(\alpha^p) - \phi_M^0(\alpha^p) \tag{3.13}$$

Theorem 3.4 (Theorem (2.4) [Alm62])**.** *There exists a positive number* ε_M *with the following property:*

Let $f : I(m+1, 0)_0 \to \mathcal{Z}_k(M)$ *be any homomorphism satisfying* $\mathcal{F}_M(f(\alpha), f(\beta)) < \varepsilon_M$ *whenever* α *and* β *are* 0*-cells in the vertex set of some* 1*-cell in* $I(m + 1, 0)$*, whose maximum is denoted by* Θ*. Then one can find a chain map*

$$\phi_M : I(m + 1, 0) \to \mathbf{I}_*(M) \tag{3.14}$$

of degree k *such that*

(1) $\phi_M|_{I(m+1,0)_0} = f$

(2) *For each* $\alpha \in I(m + 1, 0)_p (p \ge 1)$, $\phi_M(\alpha)$ *is an isoperimetric choice for* $\phi_M(\partial \alpha)$

(3) *For each* $\alpha \in I(m + 1, 0)_p (p \ge 1)$, $\mathbf{M}(\phi_M(\alpha)) \le \Theta$

(4) *If ϕ'_A is another chain map satisfying the above conclusions, then ϕ'_A is chain homotopic with ϕ_A*

Proof. The theorem follows directly from the **Proposition 3.1**. □

3.3. The Constrution of the Map F_M.

For each homotopy class $[f] \in \pi_k(\mathcal{Z}_m(M); 0)$, choose a representative map $f : (I^k, \partial I^k) \to (\mathcal{Z}_m(M), 0)$. Apparently, f induces a chain map of degree k

$$\phi_M : (I(k,n), \partial I(k,n)) \to (\mathbf{I}_*(M), 0) \tag{3.15}$$

as long as $n \geq N_f$ where $\mathcal{F}(f(u), f(v)) \leq \varepsilon_M$ whenever $\mathrm{dist}(u, v) \leq 2^{-N_f}$.

If $\alpha_1, \cdots, \alpha_p$ are the k-cells of $I(k,n)_m$, then

$$\sum_{i=1}^{p} \phi_M(\alpha_i) \tag{3.16}$$

is a cycle in $\mathbf{I}_{m+k}(M)$.

Thus, we can define

$$F_M([f]) := \left[\sum_{i=1}^{p} \phi_M(\alpha_i)\right] \tag{3.17}$$

Exercise 6. *F_M is well-defined.*

3.4. The Construction of the Map E_M and the surjectiveness of F_M.

Now we would like to seek a map $E_M : H_{m+k}(M) \to \pi_k(\mathcal{Z}_m(M); 0)$ such that

$$F_M \circ E_M = \mathrm{Id}_{H_{m+k}(M)} \tag{3.18}$$

This would imply that F_M is an epimorphism.

Note that M is a smooth closed Riemannian manifold, and there exists a Lipschitz retraction $r : U := B_\eta(M) \to M$ for some small $\eta > 0$. For any homology class $\tau \in H_{m+k}(M)$, by deformation theorem, we can always find an integral polyhedral chain

$$T \in \mathcal{Z}_{m+k}(U) \tag{3.19}$$

such that the homology class of $r_\#(T) \in \mathcal{Z}_{m+k}(M)$ is τ. W.l.o.g., we may also assume that

$$\mathrm{spt}(T) \subset \mathbb{R}^L \cap \{x_i \in (0,1), \forall i\} \tag{3.20}$$

and no coordinate vector $(0, \cdots, 0, 1, 0, \cdots, 0)$ of $\mathbb{R}^L$ is parallel with any face of T of positive dimension.

Define $f_1 : (I, \partial I) \to (\mathcal{Z}_{m+k-1}(U), 0)$ to be

$$f_1(t) := \partial(T \cap \{x : x_1 < t\}) \tag{3.21}$$

And then inductively, define $f_i : (I^i, \partial I^i) \to (\mathcal{Z}_{m+k-i}(U), 0)$ for $i = 2, \cdots, k$ as

$$f_i(t_1, t_2, \cdots, t_i) := \partial(f_{i-1}(t_{i-1}) \cap \{x : x_i < t_i\}) \tag{3.22}$$

For $(t_1, t_2, \cdots, t_k) \in I^k$, we can define a continuous map $f := r_\# \circ f_k$ and set

$$E_M(\tau) = [f] \tag{3.23}$$

It is straightforward to show that (even though we haven't proven the well-defineness of E_M)

$$F_M \circ E_M(\tau) = \tau \tag{3.24}$$

3.5. **Injectiveness of F_M and Other Consequences.**

The proof of the injectiveness of F_M would be as follows: Suppose

$$[f] \in \pi_k(\mathcal{Z}_m(M); 0) \quad F_M([f]) = [0] \tag{3.25}$$

and then we only need to seek a formula for a homotopy

$$\begin{aligned} h &: ([0,3] \times I^m, [0,3] \times \partial I^m) \to (\mathcal{Z}_k(M), 0) \\ h(0,x) &= 0 \quad \forall x \in I^m \\ h(3,x) &= f(x) \quad \forall x \in \partial I^m \end{aligned} \tag{3.26}$$

The formula was constructed by hand in [Alm62] which is too complicated to be written down here. The key of the construction lies in finding the interpolation formula (Section 6 [Alm62]) for chain maps. By applying the interpolation formula on chain homotopies between different chain maps induced from the same f, one could obtain a homotopy between f and 0 if $F_M([f]) = [0]$.

In fact, the same results hold for $\mathcal{Z}_k(M; \mathbb{Z}_2)$, $\mathcal{Z}_k(M; \mathbf{M})$ and $\mathcal{Z}_k(M; \mathbf{M}; \mathbb{Z}_2)$ as well.

Theorem 3.5. *For any smooth closed Riemannian manifold M and any nonnegative integers k and m, we have*

$$\begin{aligned} \pi_k(\mathcal{Z}_m(M; \mathbf{M}); 0) &\cong H_{k+m}(M) \\ \pi_k(\mathcal{Z}_m(M; \mathbb{Z}_2); 0) &\cong H_{k+m}(M; \mathbb{Z}_2) \\ \pi_k(\mathcal{Z}_m(M; \mathbf{M}; \mathbb{Z}_2); 0) &\cong H_{k+m}(M; \mathbb{Z}_2) \end{aligned} \tag{3.27}$$

4. Existence of Almost Minimizing Varifolds

4.1. **Abstract Homotopy Relations.**

Definition 4.1 (Cell complexes)**.**

(1) *Define $I_0(m,j)_0$ to be the set of 0-cells on ∂I^m.*
(2) *Define $d : I(m,j)_0 \times I(m,j)_0 \to \mathbb{N}$ to be*

$$d(x,y) = 3^j \sum_{i=1}^m |x_i - y_i| \tag{4.1}$$

and similarly $d_\infty(x,y) = 3^j \sup |x_i - y_i|$.
(3) *Whenever $\varphi : I(m,j)_0 \to \mathcal{Z}_n(M^{n+1}; \mathbf{M}; \mathbb{Z}_2)$, we define the **fineness** of φ by*

$$f(\varphi) = \sup\{d(x,y)^{-1} \mathbf{M}(\varphi(x), \varphi(y))\} \tag{4.2}$$

(4) *For $i,j \in \mathbb{N}$, we deine $n(i,j) : I(m,i)_0 \to I(m,j)_0$ to be*

$$n(i,j)(x) = \operatorname{argmin}_{y \in I(m,j)_0} d(x,y) \tag{4.3}$$

Now we can define the homotopy maps.

Definition 4.2 (Homotopy maps)**.**

(1) *Let $\delta > 0$. We say that φ_1 **is** m **homotopic to** φ_2 **with fineness** δ if and only if there exist positive integers k_1, k_2, k_3 and a map*

$$\psi : I(1,k_3)_0 \times I(m,k_3)_0 \to \mathcal{Z}_n(M; \mathbf{M}; \mathbb{Z}_2) \tag{4.4}$$

such that $f(\psi) < \delta$ and $f(\varphi_j) < \delta$, and whenver $j = 1, 2$ and $x \in I(m, k_3)_0$,

$$\begin{aligned} \varphi_j : I(m, k_j) \to \mathcal{Z}_n(M; \mathbf{M}; \mathbb{Z}_2), \quad \varphi_j[I_0(m, k_j)_0] = 0 \\ \psi([j-1], x) = \varphi_j(n(k_3, k_j)(x)), \quad \psi[I(1, k_3)_0 \times I_0(m, k_3)_0] = 0 \end{aligned} \tag{4.5}$$

(2) *An m* ***homotopy sequence of mappings*** *is a sequence of mappings $\{\varphi_i\}$ for which there exist positive numbers $\delta_i \to 0$ such that φ_i is m homotopic to φ_{i+1} with fineness δ_i and*

$$\sup\{\mathbf{M}(\varphi_i(x)) : x \in \mathrm{dmn}(\varphi_i), \forall i \in \mathbb{N}^+\} < \infty \tag{4.6}$$

(3) *Two m homotopy sequences of mappings $S_1 = \{\varphi_i\}$ and $S_2 = \{\psi_i\}$ are* ***homotopic*** *if there exist positive numbers $\delta_i \to 0$ such that φ_i is m homotopic to ψ_i with fineness δ_i.*

(4) *A nonempty collection of m homotopy sequences of mappings Π is called an m* ***homotopy class of mappings*** *if*
- *$S_2 \in \Pi$ whenever S_1 and S_2 are homotopic and $S_1 \in \Pi$.*
- *All $S \in \Pi$ are homotopic to each other.*

Definition 4.3 (Min-max Sequences)**.**

(1) *Define $\mathbf{L} : \Pi \to \mathbb{R}$ to be*

$$\mathbf{L}(\{\varphi_i\}) = \limsup_{i\to\infty} \sup\{\mathbf{M}(\varphi_i(x))\} \tag{4.7}$$

(2) *Define the* ***min-max width*** *of Π by*

$$\mathbf{L}(\Pi) = \inf\{\mathbf{L}(S) : S \in \Pi\} \tag{4.8}$$

(3) *$S \in \Pi$ is called a* ***min-max sequence*** *of Π provided that $\mathbf{L}(\Pi) = \mathbf{L}(S)$.*

(4) *If S is a min-max sequence, then its* ***critical set*** *is defined to be*

$$\mathbf{C}(S) = \{V \in \mathcal{V}_n(M) : V = \lim_j |\varphi_{i_j}(x_j)|, \quad \|V\|(M) = \mathbf{L}(S)\} \tag{4.9}$$

Note that $\mathbf{C}(S)$ is nonempty and compact.

Exercise 7. *Every m homotopy class of mappings contains a min-max sequence. (Hint: use the diagnal method.)*

In the following, we will denote the collection of m homotopy classes of mappings by

$$\pi_m^{\#}(\mathcal{Z}_n(M^{n+1}; \mathbf{M}; \mathbb{Z}_2), 0) \tag{4.10}$$

And the Almgren's isomorphism also holds, i.e.,

$$\pi_m^{\#}(\mathcal{Z}_n(M^{n+1}; \mathbf{M}; \mathbb{Z}_2), 0) \cong H_{m+n}(M^{n+1}; \mathbb{Z}_2) \tag{4.11}$$

Exercise 8. *There exists $\Pi \in \pi_1^{\#}(\mathcal{Z}_n(M^{n+1}; \mathbf{M}; \mathbb{Z}_2), 0)$ such that $[\Pi] \neq 0 \in H_{n+1}(M^{n+1}; \mathbb{Z}_2)$ and moreover, the width $\mathbf{L}(\Pi) > 0$. (Hint: Use isoperimetric theorem)*

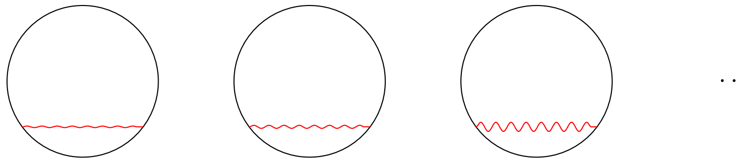

FIGURE 2. Wiggling curves

4.2. **Pull-tight.**

Let's recall the min-max construction by Birkhoff for closed geodesics in a sphere and suppose that $\{\gamma_i(t)\}$ is a min-max sequence of sweepouts. Before the curve shortening process, a sequence of curves $\{\gamma_i(t_i)\}$ might not converge to a closed geodesic as **Figure 2** indicates. However, the Birkhoff curve shortening process will help to reduce drastically the length of curves wiggling a lot. Thus, we can show that the sequence of shortened curves shall converge to a closed geodesic as long as their lengths converge to the width.

Similarly, in the Almgren-Pitts min-max theory, we will introduce some similar process, which intuitively pulls wiggling hypersurfaces tight.

Theorem 4.1 (Theorem 4.3 [Pit81]). *For any min-max sequence $S \in \Pi$, there exists another min-max sequence $S^* \in \Pi$ such that $\mathbf{C}(S^*) \subset \mathbf{C}(S)$ and any $V \in \mathbf{C}(S^*)$ is stationary in M.*

Outline of proof. Suppose that $S = \{\varphi_i\}$ and let $c = \sup_{i,x} \mathbf{M}(\varphi_i(x)) < \infty$.

We can define compact sets

$$\begin{aligned} A &= \mathcal{V}_k(M) \cap \|V\|(M) \leq c \\ A_0 &= A \cap \{\delta V = 0\} \\ A_1 &= A \cap \{\mathbf{F}(V, A_0) \geq 2^{-1}\} \\ A_i &= A \cap \{2^{-i} \leq \mathbf{F}(V, A_0) \leq 2^{-i+1}\}, i = 2, 3, \cdots \end{aligned} \tag{4.12}$$

Note that for any $V \in A \backslash A_0$, $\delta V \neq 0$ so we can always associate V with a vector field X_V such that $\delta V(X_V) < 0$. Therefore, by the construction of a paracompact covering and the associated partition of unity, we can construct a continuous map from $A \backslash A_0$ to the space of C^1 vector fields on M.

The one parameter groups of diffeomorphisms from the vector fields above shall pull S tight and eventually, $\mathbf{C}(S^*)$ will bypass all nonstationary varifolds.

However, there will be an issue in this pull-tight process, i.e., the fineness with respect to the $\mathbf{M}$ norm, since the M continuity would not be preserved under one parameter groups of diffeomorphisms. Therefore, we need to interpolate the maps after pulling tight.

Note that the naive way of applying Almgren's interpolation fomula could lead to pulling back the varifolds along the pulling-tight diffeomorphisms, so we could not guarantee that $\mathbf{C}(S^*)$ only consists of stationary varifolds. The correct way to do this is to use the interpolation results from discrete to continuous and from continuous to discrete in [MN14] multiple times as the **Figue 3** indicates.

□

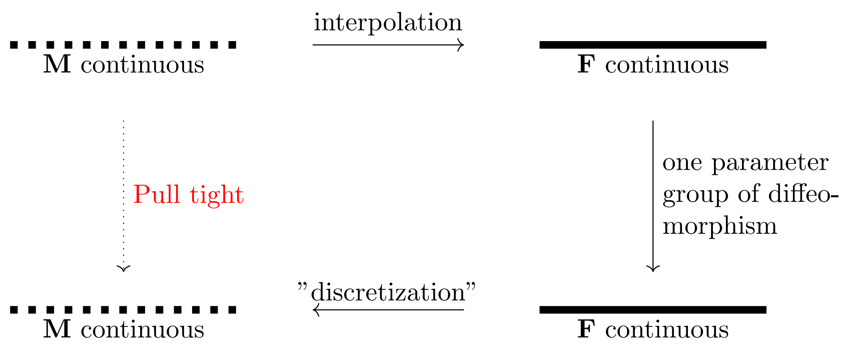

FIGURE 3. Pull tight

4.3. **Combinatorial Argument.**

Now we introduce a combinatorial argument to help establish the result that $\mathbf{C}(S)$ has at least one varifold with the almost minimizing property in **Theorem 2.4**.

Proposition 4.1. *Given $I(m,k)$, for nay $\sigma \in I(m,k)_0$. we associate it with $A(\sigma) = \{\bar{A}(p, s_j, r_j), j = 1, 2, \cdots, 3^m\}$ with $r_j > s_j > 2r^{j+1}(r_1 < \mathrm{inj}_M/2)$. Then we can find a function*

$$\alpha : I(m,k)_0 \to \cup A(\sigma) \tag{4.13}$$

such that $\alpha(\sigma) \in A(\sigma)$ and $\alpha(\sigma) \cap \alpha(\tau) = \emptyset$ whenver $\sigma \neq \tau$ and $\sigma, \tau \in \mathrm{spt}\,(\gamma)$ for some $\gamma \in I(m,k)$.

Proof. We can repeatedly take the annulus with smallest radius from the union of $A(\sigma)$ where $\alpha(\sigma)$ is not defined yet and define α at the corresponding 0-cell. Then we could discard all the annuli intersecting with it. It is easy to check that we can obtain the desired α since we only discard one annulus from each $A(\sigma)$ in each step. □

Now we can state our main existence theorem.

Theorem 4.2 (Theorem 4.10 [Pit81])**.** *If Π is an m homotopy class of maps, then for any pulled-tight min-max sequence $S \in \Pi$, there exists a varifold V in the critical set $\mathbf{C}(S)$ satisfies:*

1. *$\|V\|(M) = \mathbf{L}(\Pi)$.*
2. *V is stationary in M.*
3. *For any collection A of $J_m = 3^m$ concentric closed annuli $\{\bar{A}(p, s_i, r_i)\}$ as in **Proposition 4.1**, V is almost minimizing in at least one open $A(p, s_i, r_i)$.*

In particular, the last property implies that for each $p \in M$, there exists $r(p) > 0$ such that V is almost minimizing in $A_0(p, s, r)$ for all $s \in (0, r)$.

Proof. W.l.o.g., we may assume that $\mathbf{L}(\Pi) > 0$ and $S = \{\varphi_i\}$ is the pulled-tight min-max sequence. Suppose that the argument is not true, and for any $V \in \mathbf{C}(S)$, there exists a collection $A(V) = \{\bar{A}(p, s_i, r_i)\}_{i=1}^{J_m}$ such that V is not almost minimizing in any $A(p, s_i, r_i)$.

Recall the definition of almost minimizing property and the compactness of $\mathbf{C}(S)$ which indicate that there exist a finite sequence $\{V_i\}_{i=1}^M \subset \mathbf{C}(S)$ and an associated sequence of positive numbers $\{\varepsilon_i\}_{i=1}^M$ such that

(1) $\mathbf{C}(S) \subset \bigcup_{i=1}^M \{V \in \mathcal{V}_n(M) : \mathbf{F}(V, V_i) < 4^{-1}\varepsilon_i\}$.
(2) For each V_i, $T \in \mathcal{Z}_n(M; \mathbf{M}; \mathbb{Z}_2)$ with $\mathbf{F}(|T|, V_i) < \varepsilon_i$ and any $\delta > 0$, $j = 1, 2, \cdots, J_m$, there exits a sequence $T = T_1, T_2, \cdots, T_q \subset \mathcal{Z}_n(M; \mathbf{M}; \mathbb{Z}_2)$ such that

$$\begin{aligned} &\bigcup_k \operatorname{spt}(T_k - T) \subset A(p^i, \tilde{s}^i_j, \tilde{r}^i_j) \\ &\sup_k \mathbf{M}(T_k - T_{k-1}) \le \delta \\ &\sup_k \mathbf{M}(T_k) \le \mathbf{M}(T) + \delta \\ &\varepsilon_k < \mathbf{M}(T) - \mathbf{M}(T_q) \end{aligned} \tag{4.14}$$

where $\tilde{s}^i_j < s^i_j < r^i_j < \tilde{r}^i_j$ satisfying the condition that $\tilde{s}^i_j > 2\tilde{r}^i_{j+1}$.

We also denote $\min(\min\{2s^i_j - r^i_{j+1}\}/100, 1) > 0$ by t for convenience.

We can choose small $\varepsilon \in (0, \min_i\{\frac{\varepsilon_i}{2}\})$ and large $N \in \mathbb{N}$ such that $\forall i \ge N$,

(1) Either $\mathbf{M}(\varphi_i(x)) < \mathbf{L}(S) - 2\varepsilon$, or $\mathbf{F}(|\varphi_i(x)|, V_j) < \frac{1}{2}\varepsilon_j$ for some $j = 1, 2, \cdots, M$.
(2) $mf_{\mathbf{M}}(\varphi_i) \le t\varepsilon/(2 \cdot 6^m)$.
(3) For any $T, S \in \mathcal{Z}_n(M; \mathbf{M}; \mathbb{Z}_2)$, if $\mathbf{M}(T - S) \le \varepsilon$, then there exists $Q \in \mathbf{I}_{n+1}(M; \mathbb{Z}_2)$ with $\partial Q = T - S$ and

$$\mathbf{M}(Q) \le \mathbf{M}(T - S) \tag{4.15}$$

We now show that for any fixed $i > N$, $\sup_x \mathbf{M}(\varphi_i(x))$ can be decreased by at least $\varepsilon > 0$ along a discrete homotopy without increasing $f_{\mathbf{M}}$ too much. Then the new sequence $S^* \in \Pi$ shall give a contradiction, since

$$\mathbf{L}(\Pi) \le \mathbf{L}(S^*) \le \mathbf{L}(S) - \varepsilon \le \mathbf{L}(\Pi) - \varepsilon \tag{4.16}$$

Fixed $i > N$, whenever $x \in I(m, n_i)_0 = \operatorname{dmn}(\varphi_i)$ with $\mathbf{M}(\varphi_i(x)) \ge \mathbf{L}(S) - 2\varepsilon$, we choose V_{i_x} such that

$$\mathbf{F}(|\varphi_i(x)|, V_{i_x}) < \frac{\varepsilon_{i_x}}{2} \tag{4.17}$$

and by **Proposition 4.1**, we know that we can associate such x with $A(p_x, \tilde{s}_x, \tilde{r}_x) \in A(V_{i_x})$. The set of all such x will be denoted by $\mathcal{B}_i$.

In order to construct a homotopy, we need 2 ingredients.

The first ingredient is an application of the slicing theorem [Sim84]. For $\sigma_0 \in I(m, n_i)$, $x, y \in \operatorname{spt}\sigma_0$ $(x \ne y)$ and $x \in \mathcal{B}_i$, by Isoperimetric Theorem, there exists $Q \in \mathbf{I}_{n+1}(M; \mathbb{Z}_2)$ with $\partial Q = \varphi_i(x) - \varphi_i(y)$ and $\mathbf{M}(Q) \le mf_{\mathbf{M}}(\varphi_i) \le t\varepsilon/(2 \cdot 6^m)$.

if we write $u(x) = |x - p_x|$ for $x \in M$, we have

$$\int_{\tilde{s}_x - t}^{*\tilde{r}_x + t} \mathbf{M}\langle Q, u, s+\rangle \mathrm{d}\mathcal{L}^1(s) \le \mathbf{M}(Q) \tag{4.18}$$

Therefore, there exists $\tilde{\tilde{s}}_x < \tilde{s}_x < \tilde{r}_x < \tilde{\tilde{r}}_x$ such that

$$\begin{aligned}&\|\varphi_i(x_j)\|\partial B(p,\tilde{\tilde{s}}_x)\cup\partial B(p,\tilde{\tilde{r}}_x)=0\\&\langle Q,u,\tilde{\tilde{s}}_x+\rangle\in\mathbf{I}_n(M;\mathcal{Z}_2)\\&\langle Q,u,\tilde{\tilde{r}}_x+\rangle\in\mathbf{I}_n(M;\mathcal{Z}_2)\\&\mathbf{M}\langle Q,u,\tilde{\tilde{s}}_x+\rangle\le\varepsilon/2^{m+1}\\&\mathbf{M}\langle Q,u,\tilde{\tilde{r}}_x+\rangle\le\varepsilon/2^{m+1}\end{aligned}\tag{4.19}$$

Note that here $\tilde{\tilde{s}}_x$ and $\tilde{\tilde{r}}_x$ can be chosen to be independent of σ and y with the same estimates.

The other ingredient is a restatement of the non-almost-minimizing property. By assumption, let $\delta = f_{\mathbf{M}}(\varphi_i)$ and we also know that there exists $N_1 = N_1(i) \in \mathbb{N}$ such that $\forall x \in \mathcal{B}_i$, there exits a sequence $\varphi_i(x) = T_1^x, T_2^x, \cdots, T_{3^{N_1}}^x \subset \mathcal{Z}_n(M;\mathbf{M};\mathbb{Z}_2)$ such that

$$\begin{aligned}&\bigcup_k \operatorname{spt}(T_k^x - T)\subset A(p_x,\tilde{s}_x,\tilde{r}_x)\\&\sup_k \mathbf{M}(T_k^x - T_{k-1}^x)\le\delta\\&\sup_k \mathbf{M}(T_k^x)\le\mathbf{M}(T)+\delta\\&\varepsilon_{i_x}<\mathbf{M}(\varphi_i(x))-\mathbf{M}(T_{3^{N_1}}^x)\end{aligned}\tag{4.20}$$

In this case, we shall define the replacement map

$$R:\{(x,y)\in\mathcal{B}_i\times I(m,n_i)_0|\exists\sigma\in I(m,n),x,y\in\sigma\}\to\mathcal{Z}_n(M;\mathbf{M};\mathbb{Z}_2)\tag{4.21}$$

by

$$\begin{aligned}R(x,y)=\varphi_i(y)\llcorner(M\backslash\bar{B}(p_x,\tilde{\tilde{r}}_x))\cup B(p_x,\tilde{\tilde{s}}_x)+\varphi_i(x)\llcorner A_0(p_x,\tilde{\tilde{s}}_x,\tilde{\tilde{r}}_x)\\-\langle Q,u,\tilde{\tilde{s}}_x+\rangle+\langle Q,u,\tilde{\tilde{r}}_x+\rangle\end{aligned}\tag{4.22}$$

and verify that

$$\begin{aligned}\mathbf{M}(R(x,y)-\varphi_i(x))\le 2\cdot 3^m f_{\mathbf{M}}(\varphi_i)/t\\\mathbf{M}(R(x,y)-\varphi_i(y))\le 2\cdot 3^m f_{\mathbf{M}}(\varphi_i)/t\end{aligned}\tag{4.23}$$

For convenience, we extend R to $\mathcal{B}_i \times I(m,n_i)_0$ with 0 value.

Now, let $N_2 = n_i + N_1 + 2$ and we can define the homotopy (visualized in **Figure 4**)

$$\psi: I(0,N_2)_0\times I(m,N_2)_0\to\mathcal{Z}_k(M;\mathbf{M};\mathbb{Z}_2)\tag{4.24}$$

by

(1) $\psi(0,y) = \varphi_i(0, n(N_2,n_i)(y))$
(2) $\psi(3^{-N_2},y) = \sum_{x\in\mathcal{B}_i, d_\infty(n(x),y)\le 8\cdot 3^{N_2}} (R(x,n(N_2,n_i)(y)) - \varphi_i(0,n(N_2,n_i)(y))) + \varphi_i(0,n(N_2,n_i)(y))$
(3) $\psi(k\cdot 3^{-N_2},y) = \sum_{x\in\mathcal{B}_i, d_\infty(n(x),y)\le 8\cdot 3^{N_2}} (T^x_{\min\{3^{N_2},k,8\cdot 3^{N_2}-d_\infty(x,y)\}} - \varphi_i(x))\llcorner A_0(p_x,s_x,r_x) + \psi(3^{-N_2},y)$ for $k \ge 1$.

where $n(x)$ is an abbreviation for $n(n_i,N_2)(x)$.

We can verify that $\varphi_i^* := \psi(1,\cdot)$ satisfies that $\sup\varphi_i^* \le \sup\varphi_i - \varepsilon$ and $S^* = \{\varphi_i^*\}$ gives the contradiction.

□

FIGURE 4. Homotopy Construction

5. REGULARITY

In this section, we shall show that the support of a varifold with almost minimizing property as in **Theorem 2.4** is smooth outside a singular set of codimension no less than $n-7$ and then the existence result in **Theorem 4.2** will conclude the existence of one minimal hypersurface in any closed manifold.

Theorem 5.1 (Theorem A [Pit81],Theorem 4 [SS81])**.** *For any smooth closed manifold $M^{n+1}(n \geq 2)$, there exists a smooth minimal hypersurface $\Sigma \subset M$ with $\mathcal{H}^\alpha(\operatorname{sing}(\Sigma)) = 0$ for any non-negative $\alpha > n-7$. In particular in case $n \leq 6$, Σ is smooth.*

The regularity argument was originally proved by R. Schoen and L. Simon [SS81] (Pitts proved it for the case $2 \leq n \leq 5$ using Schoen-Simon-Yau curvature estimate), where they derived the following compactness theorem for stable minimal hypersurfaces.

Theorem 5.2 (Theorem 2 [SS81])**.** *Suppose $\{\Sigma_i\}$ is a sequence of orientable C^2 open hypersurface with*

$$0 \in \overline{\Sigma}_i, \mathcal{H}^{n-2}\left(\left(\overline{\Sigma}_i \backslash \Sigma_i\right) \cap B^{n+1}(0,\rho_0)\right) = 0 \tag{5.1}$$

Suppose that each Σ_i is stable in $B^{n+1}(0,\rho_0)$ and $\limsup_{i\to\infty} \mathcal{H}^n(M_i \cap B^{n+1}(0,\rho_0)) < \infty$. Then there exist a subsequence, still denoted by $\{\Sigma_i\}$, and a varifold V such that

$$V = \lim_i |M_i \cap B^{n+1}(0, \frac{1}{2}\rho_0)| \tag{5.2}$$

Moreover,

$$\operatorname{spt}\|V\| \cap B(0, \frac{1}{2}\rho_0) = \overline{\Sigma} \cap B(0, \frac{1}{2}\rho_0) \tag{5.3}$$

where Σ is an orientable hypersurface with $\mathcal{H}^n(\Sigma \cap B(0, \frac{1}{2}\rho_0)) < \infty, 0 \in \overline{\Sigma}$ and $\mathcal{H}^\alpha(\operatorname{sing}(\Sigma) \cap B(0, \frac{1}{2}\rho_0)) = 0$ for any non-negative $\alpha > n-7$.

Recall that **Theorem 4.2** and **Theorem 2.4** imply that there exists a nonempty set $\mathfrak{A} \subset \mathcal{IV}_n(M)$ consisting of non-zero stationary integral varifolds V with the properties that for any $p \in M \cap \operatorname{spt}\|V\|$, there is an $r(p) > 0$ such that for each $0 < s < t < r(p)$, there exists a varifold $V^* \in \mathfrak{A}$ (a replacement or a comparison surface) with

(1) $\|V^*\|(M) = \|V\|(M)$
(2) $V^* \llcorner G_n(M \backslash \bar{A}(p,s,t)) = V \llcorner G_n(M \backslash \bar{A}(p,s,t))$
(3) $V^* \llcorner G_n(A(p,s,t)) = (\lim_{i\to\infty} |T_i|) \llcorner G_n(A(p,s,t))$, where

$$T_j \in \mathcal{Z}_n(M, M\backslash A(p,s,t); \mathbb{Z}_2) \tag{5.4}$$

with $\sup \mathbf{M}(T_j) < \infty$ is locally area minimizing in $A(p,s,t)$ and such that $|T_j|$ is stable in $A(p,s,t)$. In particular, the regularity result by Federer

[Fed70] implies that $\operatorname{spt}T_j \cap A(p,s,t)$ is a smooth minimal hypersurface outside a singular set of codimension no less than 7.

By virtue of (3) and the compactness reuslt (**Theorem 5.2**), we see that $\operatorname{spt}\|V^*\| \cap A(p,s,t) = \bar{\Sigma} \cap A(p,s,t)$ is also a smooth minimal hypersurface outside a singular set of codimension no less than 7.

Firstly, we show that $C \in \operatorname{Var}\operatorname{Tan}(V,p)$ has the property that $C \in \mathcal{IV}_n(\mathbb{R}^L)$ with $\mathcal{H}^\alpha(\operatorname{sing}(C)) = 0$ for any nonnegative $\alpha > n-7$.

W.l.o.g., we may assume that

$$\mu_{t_q^{-1}\#} \circ \tau_{p\#} V \to C \in \mathcal{V}_n(\mathbb{R}^L) \tag{5.5}$$

where $t_q \searrow 0 (t_q < r(p)/4)$, μ is the dilation map and τ is the translation map. As a consequence, $C \neq \emptyset$ and it is a stationary integral cone. The replacement property (3) gives V_q^* for the region $A(p,t_q,2t_q)$ and $W := \lim_{q\to\infty} \mu_{t_q^{-1}\#} \circ \tau_{p\#} V_q^* \in \mathcal{IV}_n(\mathbb{R}^L)$ with

$$\mathcal{H}^\alpha(\operatorname{sing} W \cap A(0,1,2)) = 0, \forall \alpha \geq 0, \alpha > n-7 \tag{5.6}$$

Note that W and C coindice in $G_n(\mathbb{R}^L \cap \bar{A}(0,1,2))$, and W is stationary as well. Since

$$\lim_{R\to\infty} R^{-n}\|W\|(B(0,R)) = \lim_{R\to\infty} R^{-n}\|C\|(B(0,R)) = \omega_n\Theta^n(\|C\|,0) = \omega_n\Theta^n(\|W\|,0) \tag{5.7}$$

and the monotonicity formula implies that $\rho^{-n}\|W\|(B(0,\rho)) \equiv \omega_n\Theta^n(\|W\|,0)$ for any $\rho > 0$, so W is also a rectifiable cone and $W = C$. Thus, we conclude that $\operatorname{spt}C$ is smooth outside a small singular set.

Now, we are going to show that V itself is regular, which heavily relies on the replacement property. Let V^* be a replacement for V in $A(p,s,t)$ and

Recall that the stationarity (Proposition 2.5 [Pit81]) implies that $\forall p \in M, \exists t \in (0, r(p))$ whenever $\tau \in (0,t)$

$$\emptyset \neq \operatorname{spt}\|F\| \cap \partial B(p,\rho) = \partial B(p,\rho) \cap \overline{(\operatorname{spt}\|F\| \backslash \bar{B}(p,\rho))} \tag{5.8}$$

for any $\rho \in (\tau, t)$ and $F \in \mathcal{IV}_n(M)$ stationary and nonzero in $A(p,\tau,t)$.

By Sard's theorem, let $0 < s_1 < s < s_2 < t$ with $\partial B(p,s_2) \pitchfork (\operatorname{spt}\|V^*\|)$, and let V^{**} be a replacement for V^* on $A(p,s_1,s_2)$.

Let $Y_0 \in \operatorname{reg}\|V^*\| \cap \partial B(p,s_2)$, $\sigma > 0$ small enough and let

$$\begin{aligned} \Sigma &= B(Y_0,\sigma) \cap \partial B(p,s_2) \\ \Sigma_+ &= B(Y_0,\sigma) \cap B(p,s_2) \\ \Gamma &= B(Y_0,\sigma) \cap \partial B(p,s_2) \cap \operatorname{reg}\|V^*\| \end{aligned} \tag{5.9}$$

Fix $X \in \Gamma$ and $e_X \in \mathbb{R}^L$ such that $e_X \in (T_X\Sigma)^\perp$ points into $B(p,s_2)$. We start with showing that

$$\lim_{k\to\infty} \mu_{\lambda_k^{-1}\#}\tau_{X\#}V^{**} = \Theta^n(\|V^{**}\|,X)|T_X\operatorname{reg}\|V^*\|| \tag{5.10}$$

for any $\lambda_k \to 0$.

Denote by C a tangent cone V^{**} at X, which is stationary, so we have

$$\int_{S^{L-1}\cap(\operatorname{reg}(C))_0} \langle Z, e_X\rangle \mathrm{d}\mathcal{H}^{n-1}(Z) = 0 \tag{5.11}$$

for any component $(\operatorname{reg}(C))_0$.

Since V^* and V^{**} coincide in $A(p, s_2, t)$, C and $T_X\text{reg}\|V^*\|$ have the following property:

$$\text{reg}(C) \cap \{Z : Z \cdot e_X < 0\} = T_X\text{reg}\|V^*\| \cap \{Z : Z \cdot e_X < 0\} \tag{5.12}$$

Hence, we can derive from the unique continuation property as well as the regularity of both C and $T_X\text{reg}\|V^*\|$ that

$$T_X\text{reg}\|V^*\| \subset \text{reg}(C) \tag{5.13}$$

and either $\text{reg}(C) = T_X\text{reg}\|V^*\|$ or there is another component $(\text{reg}(C))_0$ consisting entirely of points Z such that $Z \cdot e_X \geq 0$. In the latter case, (5.11) implies that the component is contained in $T_X M \cap \{Z : Z \cdot e_X = 0\} \equiv T_X\Sigma$, thus, $(\text{reg}(C))_0 \equiv T_X\Sigma$ contradicting the smallness of the singular set. Thus, $\text{reg}(C) = T_X\text{reg}\|V^*\|$ and the conclusion holds.

In order to use the unique continuation theorem, we need to represent $\text{reg}\|V^*\|$ and $\text{reg}\|V^{**}\|$ as graphs over $T_X\text{reg}\|V^*\|$ near X. This could be proved by showing that

$$\lim_{Z \to M, Z \in \text{spt}\|V^{**}\|} \frac{|(Z - X) \cdot \nu(X)|}{|Z - X|} = 0 \tag{5.14}$$

is locally uniform for X in a compact subset of Γ where $\nu(X)$ is a unit normal for $\text{spt}\|V^*\|$. Indeed, this is true due to the compactness theorem.

Then, the unique continuation theorem indicates that $\text{reg}\|V^*\| = \text{reg}\|V^{**}\|$ on $G_n(A(p, s, s_2))$.

We can repeat the argument above with another replacement $\tilde{V}^{**}$ for V^* on $A(p, \tilde{s}_1, s_2)$ with arbitrary $\tilde{s}_1 \in (s_1, s)$. Then, we obtain that

$$\tilde{V}^{**} \llcorner G_n(A(p, s, s_2)) = V^* \llcorner G_n(A(p, s, s_2)) = V^{**} \llcorner G_n(A(p, s, s_2)) \tag{5.15}$$

and moreover, the unique continuation leads to

$$\tilde{V}^{**} \llcorner G_n(A(p, \tilde{s}_1, s_2)) = V^{**} \llcorner G_n(A(p, \tilde{s}_1, s_2)) \tag{5.16}$$

Hence, we can conclude from (5.8) that

$$\text{spt}\|V\| \cap \partial B(p, \tilde{s}_1) = \text{spt}\|\tilde{V}^{**}\| \cap \partial B(p, \tilde{s}_1) = \text{spt}\|V^{**}\| \cap \partial B(p, \tilde{s}_1) \tag{5.17}$$

By the arbitrariness of $\tilde{s}_1$, we have

$$\text{spt}\|V\| \cap A(p, s_1, s) = \text{spt}\|V^{**}\| \cap A(p, s_1, s) \tag{5.18}$$

which is smooth outside a small singular set.

In sum, V itself is smooth outside a singular set of codimension no less than 7.

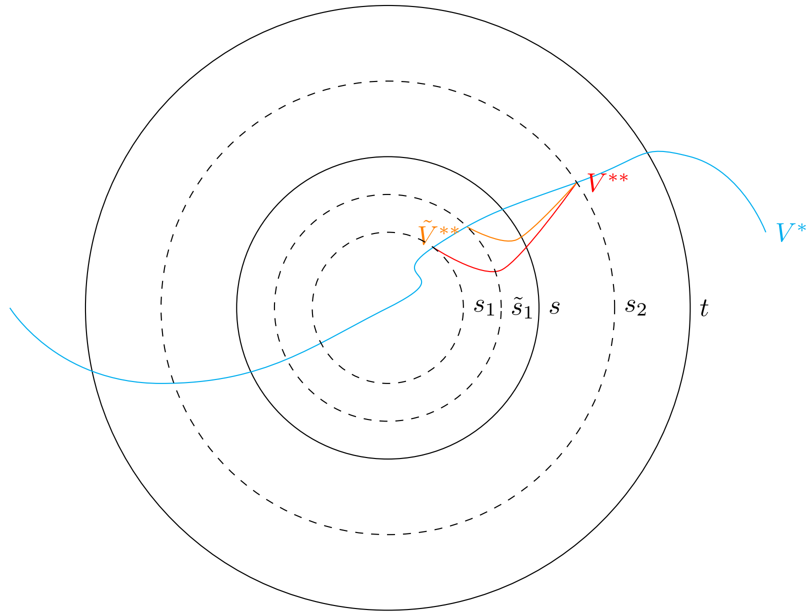

FIGURE 5. Replacement

6. Existence of Infinitely Many Minimal Hypersurfaces

In this section, we shall outline the proof for the existence of inifintely many minimal hypersurfaces in a closed manifolds with C^∞ generic metrics (Baire sense).

Given a smooth closed manifold M^{n+1}, suppose that $\mathcal{M}$ is the space of C^∞ Riemannian metrics on M and $\mathcal{M}_f \subset \mathcal{M}$ is the subset of metrics which don't admit infinitely many minimal hypersurfaces. The main theorem will be as follows.

Theorem 6.1 (Main Theorem)**.** *$\mathcal{M}_f$ is a meagre set.*

6.1. Weak Homotopic equivalence $\mathcal{Z}_n^0(M^{n+1};\mathbb{Z}_2) \sim \mathbb{RP}^\infty$ and p-width.

Recall that Almgren's Isomorphism shows that

$$\pi_k(\mathcal{Z}_n(M^{n+1};\mathbb{Z}_2),\{0\}) \cong \begin{cases} \mathbb{Z}_2, & k=0,1 \\ 0, & \text{else} \end{cases} \tag{6.1}$$

Therefore, it suffices to construct a map between $\mathcal{Z}_n^0(M^{n+1};\mathbb{Z}_2)$ and $\mathbb{RP}^\infty$ preserving the homotopy groups.

To do so, let's fixed a Morse function $f: M \to [0,1]$, and then we can define a map $\Psi: \mathbb{RP}^\infty \to \mathcal{Z}_n^0(M^{n+1};\mathbb{Z}_2)$ as follows. For any $a = [a_0 : a_1 : a_2 : \cdots : a_k : 0 : 0 : \cdots] \in \mathbb{RP}^\infty$, we can associate a polynomial $p_a : \mathbb{R} \to \mathbb{R}$ defined by $p_a(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k$. With this, we can let

$$\Psi(x) := \partial\{p_a(f) \le 0\} \tag{6.2}$$

This map gives a weak homotopic equivalence between the two spaces. Let $\bar{\lambda} \in H^1(\mathcal{Z}_n^0(M^{n+1})$ be the nontrivial element and we can define the p**-admissible set** by

$$\mathcal{P}_p = \{\Psi : X \to \mathcal{Z}_n^0(M;\mathbf{F};\mathbb{Z}_2) | X \text{ is a finite simplicial complex}, \Psi^*(\bar{\lambda}^p) \neq 0\} \tag{6.3}$$

Note that here we use the $\mathbf{F}$ metric instead of the flat norm for the continuity, and this is due to technical reasons. One insists to use the flat norm needs to assume no mass concentration [MN14].

M. Gromov [Gro03] and L. Guth [Gut09] investigated a sequence of numbers related to $\mathcal{P}_p$ called p-width ω_p defined by

$$\omega_p(M,g) = \inf_{\Psi \in \mathcal{P}_p} \sup_{x \in \mathrm{dmn}(\Psi)} \mathbf{M}(\Psi(x)) \tag{6.4}$$

More precisely, they proved that there exist two constants $c(M,g), C(M,g) > 0$ so that

$$c(M,g) \leq p^{-1/n}\omega_p \leq C(M,g) \tag{6.5}$$

Gromov also conjectured that ω_p should satisfy Weyl law as well, which was recently proved by Y. Liokumovich, F.C. Marques and A. Neves [LMN18].

Theorem 6.2. *There exists a positive constant $a(n)$ only depending on the dimension, such that*

$$\lim_{p\to\infty} p^{-1/n}\omega_p(M,g) = a(n)\mathrm{Vol}(M,g)^{n/(n+1)} \tag{6.6}$$

Note that almost directly, we also have

Lemma 6.1 (Lemma 2.1, [IMN18])**.** *$\omega_p(M,g)$ is locally Lipschitz w.r.t. g in the C^0 topology.*

6.2. **Almgren-Pitts Realizations $\mathcal{APR}_p$.** One natural question on the p-width is that whether they could be realized by some minimal hypersurfaces possibly with multiplicities from Almgren-Pitts min-max theory.

The difficulty lies in the fact that we don't bound the dimension of the cube of which the domain of Ψ in the p-admissible set could be viewed as a cubical subcomplex. As a consequence, Pitts' combinatorial arugment could not work well.

One way to overcome the difficulty relies on the application of bumpy metrics by Marques-Neves [MN16]. However, this method could only work for the case $3 \leq n+1 \leq 7$ due to the existence of singularies in minimal hypersurfaces.

Here, we outline another way found by the author, which involves some topological arguments [Li19].

Suppose that we've already chosen a min-max sequence $\{\Psi_i\}$ such that

$$\omega_p = \limsup_i \sup_{x \in \mathrm{dmn}(\Psi_i)} \mathbf{M}(\Psi_i(x)) \tag{6.7}$$

For each fixed Ψ_i with $\mathrm{dmn}(\Psi_i) = X_i$, we can restrict it to the k-skleteton $X_i^{(p)}$ and it is not difficult to check that $\Psi_i|_{X_i^{(p)}} \in \mathcal{P}_p$ as well. Moreover, by a general Whitney embbeding theorem, we may embed $X_i^{(p)}$ into I^{2p+1} and then find a cubical subcomplex to approximate it. The upshot will be a new min-max sequence $\{\Psi_i'\}$ whose elements have domains as a cubical subcomplex of I^{2p+1}.

Definition 6.1. *We define the **Almgren-Pitts Realization** for p-width, denoted by $\mathcal{APR}_p(M,g)$, to be the nonempty set of all integral rectifiable varifolds V satisfying*

- $\|V\|(M) = \omega_p(M,g)$.
- *V is a singular minimal hypersurface with optimal regularity.*

- *V has **property** $(2p+1)$, i.e., for any $q \in M$ and $J_{2p+1} = 3^{2p+1}$ concentric annuli $\{A(q, s_i, r_i)\}$ where $\{r_i\}$ and $\{s_i\}$ satisfy*

$$r_i > s_i > 2r_{i+1} \tag{6.8}$$

V is stable in at least one of $\{A(p, s_i, r_i)\}$.

Almgren-Pitts theory above shows that $\mathcal{APR}_p(M, g) \neq \emptyset$ for any p. Furthermore, by Sharp's compactness [Sha15], one can see that $\mathcal{APR}_p(M, g)$ is a compact set, and for varying metrics $g_i \xrightarrow{C^3} g$ and $V_i \in \mathcal{APR}_p(M, g_i)$, the varifold limit $V = \lim_i V_i \in \mathcal{APR}_p(M, g)$.

6.3. **Adapted Irie-Marques-Neves Argument.** For any open subset $U \subset M$, we define

$$\mathcal{M}_{U,p} = \{g \in \mathcal{M} | \forall V \in \mathcal{APR}_p(M, g), \, \|V\|(U) > 0\} \tag{6.9}$$

and

$$\mathcal{M}_U = \bigcup_{p=1}^{\infty} \mathcal{M}_{U,p} \tag{6.10}$$

Proposition 6.1. *$\mathcal{M}_{U,p}$ is an open subset of $\Gamma_\infty(M)$ for any open subset U, and so is $\mathcal{M}_U$.*

Proof. Given $g_0 \in \mathcal{M}_{U,p}$, we would like to show that there is an $\delta > 0$ such that $B_\delta(g_0, C^3) \cap \Gamma_\infty(M) \in \mathcal{M}_{U,p}$.

Suppose not, there will be a sequence $g_i \in \Gamma_\infty(M)$ such that $g_i \xrightarrow{C^3} g_0$ but $g_i \notin \mathcal{M}_{U,p}$. Therefore, we can choose a sequence $\{V_i\}$ such that $V_i \in \mathcal{APR}_p(M, g_i)$ but $V_i(U) = 0$. Up to a subsequence,

$$V_i \rightharpoonup V \tag{6.11}$$

where $V \in \mathcal{APR}_p(M, g_0)$.

Since U is open, $\|V\|(U) \leq \lim_{i\to\infty} \|V_i\|(U) = 0$ which gives a contradiction. □

Lemma 6.2 (Key Lemma)**.** *For any open subset $\mathcal{O}$ of $\mathcal{M}$, if $\mathcal{M}_f$ is dense in $\mathcal{O}$, then for any open subset $U \subset M$, $\mathcal{M}_U$ is dense in $\mathcal{O}$. Thus, $\mathcal{M}_f \cap \mathcal{O}$ is a meagre set inside $\mathcal{O}$.*

Proof. Fix U as an open subset of M. For any g in $\mathcal{O}$, from the denseness of $\mathcal{M}_f$, there is a $g' \in \mathcal{M}_f$ such that g' is arbitrarily close to g and the set

$$\begin{aligned} \mathcal{C}(g') = \{ & \sum_{j=1}^{N} m_j \mathrm{vol}_{g'}(\Sigma_j) : N \in \mathbb{N}, \{m_j\}_{j=1}^N \subset \mathbb{N}, \{\Sigma_j\}_{j=1}^N \\ & \text{singular minimal hypersurfaces with optimal regularity}\} \end{aligned} \tag{6.12}$$

is countable and thus has empty interior.

Let h be a smooth nonnegative function with $\mathrm{spt}(h) \subset U$ and $h(x) > 0$ for some $x \in U$. Let $g'(t) = (1 + th)g'$. Since $\mathcal{O}$ is open, there is $t_0 > 0$ arbitrarily small such that $\{g'(t) | 0 \leq t \leq t_0\} \subset \mathcal{O}$. The Weyl law and the fact that $\mathrm{Vol}(g'(t_0)) > \mathrm{Vol}(g'(0))$ imply the existence of $p = p(t_0) \in \mathbb{N}$ with $\omega_p(M, g'(t_0)) > \omega_p(M, g')$. Hence, by the continuity of $\omega_p(M, g'(t))$, there exists $t_1 \in (0, t_0)$ such that $\omega_p(M, g'(t_1)) > \omega_p(M, g')$ and $\omega_p(M, g'(t_1)) \notin \mathcal{C}(g')$.

Now it suffices to show that $g'(t_1) \in \mathcal{M}_{U,p}$.

Suppose not, we can find $V \subset \mathcal{APR}_p(M, g'(t_1))$ such that $\|V\|(U) = 0$. Note that $g'(t_1) = g'$ outside U so we have

$$\|V\|(M) = \sum_{j=1}^{N} m_j \mathrm{vol}_{g'(t_1)}(\Sigma_j) = \sum_{j=1}^{N} m_j \mathrm{vol}_{g'}(\Sigma_j) \in \mathcal{C} \tag{6.13}$$

where $\{\Sigma_j\}$ is a finite set of singular minimal hypersurfaces with optimal regularity with respect to both $g'(t_1)$ and g'. This gives a contradiction.

Finally, let $\{U_i\}$ be a countable basis of M, then $\mathcal{M}_f \cap \mathcal{O} \subset \bigcup_i (\mathcal{O} \backslash \mathcal{M}_{U_i})$ is a meagre set. $\square$

Proof of Main Theorem. Let $\mathcal{O} = \mathrm{Int}(\overline{\mathcal{M}_f})$ and it is easy to see that $\mathcal{M}_f \subset (\mathcal{M}_f \cap \mathcal{O}) \bigcup \partial(\overline{\mathcal{M}_f})$. From Lemma 6.2, we know that $\mathcal{M}_f \cap \mathcal{O}$ is meagre. Since $\partial(\overline{\mathcal{M}_f})$ is nowhere dense, $\mathcal{M}_f$ is also meagre. $\square$

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